

Fixed points of automorphisms of real algebraic curves. ^{*}

Jean-Philippe Monnier

Département de Mathématiques, Université d'Angers,
2, Bd. Lavoisier, 49045 Angers cedex 01, France
e-mail: monnier@tonton.univ-angers.fr

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Abstract

We bound the number of fixed points of an automorphism of a real curve in terms of the genus and the number of connected components of the real part of the curve. Using this bound, we derive some consequences concerning the maximum order of an automorphism and the maximum order of an abelian group of automorphisms of a real curve. We also bound the full group of automorphisms of a real hyperelliptic curve.

Introduction

In this note, a real algebraic curve X is a proper geometrically integral scheme over \mathbb{R} of dimension 1. Let g denote the genus of X ; throughout the paper we assume $g \geq 2$. A closed point P of X will be called a real point if the residue field at P is \mathbb{R} , and a non-real point if the residue field at P is \mathbb{C} . The set of real points $X(\mathbb{R})$ will always be assumed to be non empty. It decomposes into finitely many connected components, whose number will be denoted by s . By Harnack's Theorem we know that $s \leq g + 1$. If X has $g + 1 - k$ real connected components, we will say that X is an $(M - k)$ -curve. We will say that X has many real components if $s \geq g$ (see [Hu]). Topologically, each semi-algebraic connected component of $X(\mathbb{R})$ is homotopy equivalent to a circle.

An automorphism φ of X is an isomorphism of schemes of X with itself. Seeing X as a compact Klein surface, May proved in [Ma1] that the order of a group of automorphisms of X is bounded above by $12(g - 1)$. Moreover, he also proved that the maximum possible order of an automorphism of X is $2g + 2$ [Ma2].

In this paper, we will bound the number of fixed points of an automorphism of a real curve in terms of s . Using this bound, we will derive some consequences concerning

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the maximum order of an automorphism and the maximum order of an abelian group of automorphisms of a real curve. We will also bound the full group of automorphisms of a real hyperelliptic curve.

1 Preliminaries

We recall here some classical concepts and more notation that we will be using throughout this paper.

Let X be a real curve. We will denote by $X_{\mathbb{C}}$ the ground field extension of X to \mathbb{C} . The group $\text{Div}(X)$ (resp. $\text{Div}(X_{\mathbb{C}})$) of divisors on X (resp. $X_{\mathbb{C}}$) is the free abelian group generated by the closed points of X (resp. $X_{\mathbb{C}}$). The Galois group $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts on the complex variety $X_{\mathbb{C}}$ and also on $\text{Div}(X_{\mathbb{C}})$. We will always indicate this action by a bar. If P is a non-real point of X , identifying $\text{Div}(X)$ and $\text{Div}(X_{\mathbb{C}})^{\text{Gal}(\mathbb{C}/\mathbb{R})}$, then $P = Q + \bar{Q}$ with Q a closed point of $X_{\mathbb{C}}$.

Let $\text{Aut}(X)$ (resp. $\text{Aut}(X_{\mathbb{C}})$) denote the group of automorphisms of X (resp. $X_{\mathbb{C}}$). If $\varphi \in \text{Aut}(X)$ then φ extends to an automorphism $\varphi_{\mathbb{C}}$ of $X_{\mathbb{C}}$ such that $\overline{\varphi_{\mathbb{C}}(Q)} = \varphi_{\mathbb{C}}(\bar{Q})$ for any closed point Q of $X_{\mathbb{C}}$. We will denote by $\mu(\varphi_{\mathbb{C}})$ (resp. $\mu_{\mathbb{R}}(\varphi)$) the number of closed (resp. real closed) fixed points of $\varphi_{\mathbb{C}}$ (resp. φ).

Let G be a group and let a be an element of G . We denote by $|G|$ (resp. $|a|$) the order of G (resp. a).

Let $\varphi \in \text{Aut}(X)$. The image of a connected component of $X(\mathbb{R})$ through φ is again a connected component of $X(\mathbb{R})$. Let Σ_s denote the symmetric group corresponding to the group of permutations of the set of connected components of $X(\mathbb{R})$. We will denote by $\sigma(\varphi)$ the permutation induced by φ .

Assume G is a subgroup of $\text{Aut}(X)$ of order N . We denote by $G_{\mathbb{C}}$ the set $\{\varphi_{\mathbb{C}}, \varphi \in G\}$. Then $G_{\mathbb{C}}$ is a subgroup of $\text{Aut}(X_{\mathbb{C}})$ and $|G| = |G_{\mathbb{C}}|$ since clearly $|\varphi| = |\varphi_{\mathbb{C}}|$ if $\varphi \in \text{Aut}(X)$. The quotient space X/G is a real algebraic curve of genus g' and with above notation, $(X/G)_{\mathbb{C}} = X_{\mathbb{C}}/G_{\mathbb{C}}$. We denote by π (resp. $\pi_{\mathbb{C}}$) the morphism of real (resp. complex) algebraic curves $X \rightarrow X/G$ (resp. $X_{\mathbb{C}} \rightarrow (X/G)_{\mathbb{C}}$). We know that these maps are of degree N . The map $\pi_{\mathbb{C}}$ is ramified only at the fixed points of elements of $G_{\mathbb{C}} \setminus \{Id\}$ and the ramification index e_P at a closed point P of $X_{\mathbb{C}}$ verifies $e_P = |\text{Stab}(P)|$, with $\text{Stab}(P)$ the stabilizer subgroup of P in $G_{\mathbb{C}}$. We say that two closed points P, P' of $X_{\mathbb{C}}$ are equivalent under $G_{\mathbb{C}}$ if there exists φ in $G_{\mathbb{C}}$ such that $\varphi(P) = P'$. Using group theory, there are $\frac{N}{e_P}$ distinct points on $X_{\mathbb{C}}$ equivalent under $G_{\mathbb{C}}$ to P . Consequently the number $w(\pi_{\mathbb{C}})$ of branch points of $\pi_{\mathbb{C}}$ corresponds to the maximal number of inequivalent fixed points of elements of $G_{\mathbb{C}} \setminus \{Id\}$. Let Q denote a branch point of $\pi_{\mathbb{C}}$, we denote by $e(Q)$ the ramification index e_P of any ramification point P over Q . Considering now the map $\pi : X \rightarrow X/G$, we denote by $w_{\mathbb{R}}(\pi)$ (resp. $w_{\mathbb{C}}(\pi)$) the number of real (resp. non-real) branch points of π . Clearly $w_{\mathbb{R}}(\pi) + 2w_{\mathbb{C}}(\pi) = w(\pi_{\mathbb{C}})$. If Q is a real branch point of π then Q corresponds to a branch point Q' of $\pi_{\mathbb{C}}$ and $e(Q) = e(Q')$. If Q is a non-real branch point of π then Q corresponds to two conjugate branch points Q' and \bar{Q}' of $\pi_{\mathbb{C}}$ and $e(Q) = e(Q') = e(\bar{Q}')$. Let g' denote the genus of X/G . The Riemann-Hurwitz relation

now reads [Fa-K, p. 243]

$$2g - 2 = N(2g' - 2) + N \sum_{i=1}^{w(\pi_{\mathbb{C}})} \left(1 - \frac{1}{e(Q_i)}\right) \quad (1)$$

where $Q_1, \dots, Q_{w(\pi_{\mathbb{C}})}$ are the branch points of $\pi_{\mathbb{C}}$. Equivalently, we have

$$2g - 2 = N(2g' - 2) + N \sum_{i=1}^{w_{\mathbb{R}}(\pi)} \left(1 - \frac{1}{e(Q_i)}\right) + 2N \sum_{i=1}^{w_{\mathbb{C}}(\pi)} \left(1 - \frac{1}{e(Q'_i)}\right) \quad (2)$$

where $Q_1, \dots, Q_{w_{\mathbb{R}}(\pi)}$ (resp. $Q'_1, \dots, Q'_{w_{\mathbb{C}}(\pi)}$) are the real (resp. non-real) branch points of π . If moreover $G = \langle \varphi \rangle$ is cyclic, then we have [Fa-K, p. 245]

$$2g - 2 = |\varphi|(2g' - 2) + \sum_{i=1}^{|\varphi|-1} \mu(\varphi_{\mathbb{C}}^i) \quad (3)$$

2 Real and non-real fixed points of automorphisms of real curves

Before the study of the fixed points set of an automorphism of X we need to state basic facts about automorphisms of real curves.

Proposition 2.1 *Let G be a subgroup of $\text{Aut}(X)$. We denote by X' the quotient real curve X/G .*

- (i) *If Q is a real point of X' then either $\pi^{-1}(Q)$ is totally real or $\pi^{-1}(Q)$ is totally non-real.*
- (ii) *If Q is a non-real point of X' then $\pi^{-1}(Q)$ is totally non-real.*
- (iii) *If P is a real ramification point of π then $e_P = 2$.*
- (iv) *The image by π of a connected component C of $X(\mathbb{R})$ is a connected component of $X'(\mathbb{R})$ if and only if C does not contain any real ramification point of π . If not $\pi(C)$ is a compact connected semi-algebraic subset of a connected component C' of $X'(\mathbb{R})$ corresponding topologically to a closed interval of C' , in this situation $\pi(C)$ contains exactly two real branch points of π corresponding topologically to the end-points of $\pi(C)$.*
- (v) *The number of real branch points of π with real fibers is even on any connected component of $X'(\mathbb{R})$.*
- (vi) *The number of real ramification points of π is even on any connected component of $X(\mathbb{R})$.*

Proof: To prove statements (i) and (ii), it is sufficient to make the observation that an automorphism of X maps a real (resp. non-real) point onto a real (resp. non-real) point.

Let P be a real ramification point of π . Then $Q = \pi(P)$ is a real branch point of π . Let $t \in \mathcal{O}_Q$ be a local parameter. We may consider t as an element of \mathcal{O}_P via $\pi^\# : \mathcal{O}_Q \rightarrow \mathcal{O}_P$ and we recall that the ramification index e_P is defined as $v_P(t)$ where v_P is the valuation associated to \mathcal{O}_P . We have $e_P \geq 2$ since P is a ramification point of π . If u is a local parameter of \mathcal{O}_P then $t = u^{e_P}a$ with $a \in \mathcal{O}_P^*$. Consequently, π is given locally at P by $u \mapsto u^{e_P}$. If $e_P > 2$ then the fiber above a real point sufficiently near Q is not totally real or totally non-real contradicting statement (i). Hence (iii).

Statement (iv) implies statements (v) and (vi) since X' is a quotient space and, if P is a real ramification point of π then any point in the fiber $\pi^{-1}(\pi(P))$ is a real ramification point of π with ramification index equal to $e_P = 2$.

For (iv), let P be a real ramification point of π . Then $Q = \pi(P)$ is a real branch point of π . Let C (resp. C') denote the connected component of $X(\mathbb{R})$ (resp. $X'(\mathbb{R})$) containing P (resp. Q). Since $e_P = 2$ is even, C is clearly on one side of the fiber $\pi^{-1}(Q)$ i.e. $\pi(C)$ corresponds topologically to a closed interval of C' and Q is an end-point of this interval.

Conversely, let C be a connected component of $X(\mathbb{R})$ such that $\pi(C)$ is not a connected component of $X'(\mathbb{R})$. Then $\pi(C)$, corresponds topologically to a closed interval of a connected component of $X'(\mathbb{R})$. Let Q be one of the two end-points of this interval. Let $P \in \pi^{-1}(Q) \cap C$. Then e_P is even since C is clearly on one side of the fiber $\pi^{-1}(Q)$. Then P is a ramification point of π . \square

Definition 2.2 Let G be a subgroup of $\text{Aut}(X)$. Let X' be the quotient real curve X/G . We define $\text{Int}(\pi)$ as the number of connected components of the set

$$\{\pi(C), \quad C \text{ is a connected component of } X(\mathbb{R}) \text{ and} \\ \pi(C) \text{ is not a connected component of } X'(\mathbb{R})\}$$

By the previous proposition, we have $2\text{Int}(\pi) \leq w_{\mathbb{R}}(\pi)$. Let C be a connected component of $X(\mathbb{R})$. We denote by $\text{Stab}(C)$ the stabilizer subgroup of the component C in G i.e. the set of $f \in G$ such that $f(C) = C$.

From Proposition 2.1, we derive the following consequence concerning the real fixed points of an automorphism.

Lemma 2.3 Let φ be an automorphism of X of order N . The real ramification points of $\pi : X \rightarrow X/\langle \varphi \rangle$ are in 1 to 1 correspondence with the real fixed points of $\varphi^{\frac{N}{2}}$ and $\mu_{\mathbb{R}}(\varphi^i) = 0$ for any $i \in \{1, \dots, N-1\}$ such that $i \neq \frac{N}{2}$. Moreover, if a real ramification point of $\pi : X \rightarrow X/\langle \varphi \rangle$ belongs to a connected component C of $X(\mathbb{R})$ then exactly 2 real ramification points of $\pi : X \rightarrow X/\langle \varphi \rangle$ belong to C .

Proof: Let P be a real ramification point of $\pi : X \rightarrow X/\langle \varphi \rangle$. Then $e_P = 2$ by Proposition 2.1 (iii). Consequently $\text{Stab}(P)$ is a subgroup of order 2 of $\langle \varphi \rangle$ i.e. P is a real fixed point of $\varphi^{\frac{N}{2}}$. Since $\text{Stab}(P) = \langle \varphi^{\frac{N}{2}} \rangle$, it follows that $\mu_{\mathbb{R}}(\varphi^i) = 0$ for any $i \in \{1, \dots, N-1\}$ such that $i \neq \frac{N}{2}$. If P is a real fixed point of $\varphi^{\frac{N}{2}}$, then the image of the connected component

C of $X(\mathbb{R})$ containing P by the morphism of degree 2 $\pi' : X \rightarrow X/\langle\varphi^{\frac{N}{2}}\rangle$ is a closed interval of a connected component of $(X/\langle\varphi^{\frac{N}{2}}\rangle)(\mathbb{R})$ and P belongs to a fiber above one end-point of $\pi'(C)$ (see Proposition 2.1). Clearly C contains exactly 2 real ramification points of $\pi : X \rightarrow X/\langle\varphi\rangle$; P and the point in the fiber above the other end-point of $\pi'(C)$. \square

We give now a generalisation of [Kn, Th. 2.2.4].

Proposition 2.4 *Let G be a subgroup of $\text{Aut}(X)$. We denote by X' the quotient real curve X/G . Let C be a connected component of $X(\mathbb{R})$.*

- (i) *If $\pi(C)$ is a connected component of $X'(\mathbb{R})$ then $\text{Stab}(C)$ is cyclic.*
- (ii) *If $\pi(C)$ is not a connected component of $X'(\mathbb{R})$ then $\text{Stab}(C)$ is one of the following groups $\mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, the dihedral group D_n of order $2n$, $n \geq 3$.*
- (iii) *If $\pi(C)$ is not a connected component of $X'(\mathbb{R})$ and if $|\text{Stab}(C)| = 2n$ then C contains exactly $2n$ real ramification points of π equally shared between two fibers. Moreover, if $P \in C$ is a real fixed point of φ , then the other real fixed point of φ in C lies in the same fiber as P if and only if n is even.*

Proof: From [Kn, Th. 2.2.4], we know that $\text{Stab}(C)$ is either cyclic or dihedral (only cyclic if $\pi(C)$ is a connected component of $X'(\mathbb{R})$). So we only have to prove statement (iii).

We consider the case $\pi(C)$ is not a connected component of $X'(\mathbb{R})$. If $\text{Stab}(C) \simeq \mathbb{Z}/2$ the result follows from Proposition 2.1 and Lemma 2.3. Assume $\text{Stab}(C)$ is the dihedral group $D_n = \langle\sigma, \rho\rangle$, $n \geq 2$, with σ and ρ corresponding respectively to a symmetry and a rotation of order n of the regular polygon with n edges. If $(a, b, c, d) \in (\mathbb{Z}/n)^4$, we have $(\rho^a \sigma^b)(\rho^c \sigma^d) = \rho^{a+c(-1)^b} \sigma^{b+d}$. Since $\pi(C)$ is an interval of $X'(\mathbb{R})$, there is a real branch point Q of π such that C contains a real fixed point P of an element of $\text{Stab}(C)$. By Proposition 2.1, we have $e_P = 2$.

Firstly, we assume that P is a fixed point of a symmetry and without loss of generality we can assume that $\sigma(P) = P$. Hence $\pi^{-1}(Q) \cap C = \{\rho^k(P), k = 0, \dots, n-1\}$ and $\rho^k(P)$, $k \in \{0, \dots, n-1\}$, is a fixed point of $\rho^k \sigma \rho^{-k}$. If n is even, the other fixed point of σ in C , $\rho^{\frac{n}{2}}(P)$, is contained in $\pi^{-1}(Q) \cap C$ and conversely.

Secondly, we assume that n is even and that $\rho^{\frac{n}{2}}(P) = P$. For any $k \in \{0, \dots, n-1\}$, $\rho^k(P)$ is a fixed point of $\rho^k \rho^{\frac{n}{2}} \rho^{-k} = \rho^{\frac{n}{2}}$ and $\rho^k(P) \in C$, since $\rho^k \in \text{Stab}(C)$. From Lemma 2.3, we conclude that either $n = 2$ or $n = 4$. If $n = 4$, the other fixed point $\rho(P)$ of ρ^2 in C is contained in $\pi^{-1}(Q) \cap C$. If $n = 2$ the other fixed point $\sigma(P)$ of ρ in C is contained in $\pi^{-1}(Q) \cap C$. \square

2.1 Real fixed points of automorphisms of real curves

We will now study morphisms of degree 2 between M-curves.

Proposition 2.5 *Let φ be a non-trivial automorphism of order 2 of an M-curve such that $\mu_{\mathbb{R}}(\varphi) > 0$. Let X' denote the quotient curve $X/\langle\varphi\rangle$. then*

- (i) X' is an M -curve;
- (ii) all the fixed points of $\varphi_{\mathbb{C}}$ are real and $\mu(\varphi_{\mathbb{C}}) = \mu_{\mathbb{R}}(\varphi) = 2g + 2 - 4g'$ where g' denotes the genus of X' ;
- (iii) all the branch points of $\pi_{\mathbb{C}}$ are real and $w_{\mathbb{R}}(\pi) = 2\text{Int}(\pi) = 2g + 2 - 4g'$. Moreover, all these branch points are contained in a unique connected component of $X'(\mathbb{R})$. Concerning the other connected components of $X'(\mathbb{R})$, the inverse image by π of each of these is a disjoint union of 2 connected components of $X(\mathbb{R})$.
- (iv) $\sigma(\varphi)$ may be written as a product of g' disjoint transpositions.

Proof: From (3), we get $\mu(\varphi_{\mathbb{C}}) = 2g + 2 - 4g'$. Since $|\varphi| = 2$, we have $w_{\mathbb{R}}(\pi) + 2w_{\mathbb{C}}(\pi) = w(\pi_{\mathbb{C}}) = 2g + 2 - 4g'$ and $w_{\mathbb{R}}(\pi) = 2\text{Int}(\pi)$. It means that there are $\text{Int}(\pi)$ connected components of $X(\mathbb{R})$ such that the image by π of each of these components is not a connected component of $X'(\mathbb{R})$ and $\text{Int}(\pi) \leq g + 1 - 2g'$. Consequently, there are $g + 1 - \text{Int}(\pi)$ connected components of $X(\mathbb{R})$ such that the image by π of each of these components is a connected component of $X'(\mathbb{R})$ and $g + 1 - \text{Int}(\pi) \geq 2g'$. Let s' denote the number of connected components of $X'(\mathbb{R})$. Let v denote the number of connected components of $X'(\mathbb{R})$ that do not contain any real branch point of π . According to the above remarks, we have $v \geq \frac{g+1-\text{Int}(\pi)}{2} \geq g'$. Since π has at least 2 real branch points and by Harnack inequality, we get $v = g'$, $s' = v + 1$ and $\text{Int}(\pi) = g + 1 - 2g'$. Hence $w_{\mathbb{R}}(\pi) = w(\pi_{\mathbb{C}}) = 2g + 2 - 4g'$ and the statements (i), (ii) and (iii) follow. Statement (iv) is a consequence of statement (iii). \square

Let us mention two nice consequences of Proposition 2.5.

Theorem 2.6 *Let φ be a non-trivial automorphism of an M -curve. If one of the fixed points of $\varphi_{\mathbb{C}}$ is real then all are real.*

Proof: If φ has a real fixed point then $|\varphi| = 2$ by Lemma 2.3. By Proposition 2.5, the proof is done. \square

Corollary 2.7 *Let φ be an automorphism of order $N > 2$ of an M -curve. If $\pi : X \rightarrow X/\langle \varphi \rangle$ has at least one real ramification point then $\varphi_{\mathbb{C}}$ is fixed point free.*

Proof: Assume $\pi : X \rightarrow X/\langle \varphi \rangle$ has at least one real ramification point. By Lemma 2.3, N is even, the real ramification points of $\pi : X \rightarrow X/\langle \varphi \rangle$ are the real fixed points of $\varphi^{\frac{N}{2}}$ and $\mu_{\mathbb{R}}(\varphi^i) = 0$ for any $i \in \{1, \dots, N-1\}$ such that $i \neq \frac{N}{2}$. Let P be a fixed point of $\varphi_{\mathbb{C}}$. Since $N > 2$, P is not a real point. Clearly P is also a fixed point of $\varphi_{\mathbb{C}}^{\frac{N}{2}}$, which contradicts Theorem 2.6. \square

We state now a generalization of Theorem 2.6.

Theorem 2.8 *Let φ be a non-trivial automorphism of order N of a real curve. If $\pi : X \rightarrow X/\langle\varphi\rangle$ has at least one real ramification point then $\mu(\varphi_{\mathbb{C}}) \leq 2 + \frac{2g-2s+\mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})}{|\varphi|-1}$. Consequently, $\mu(\varphi_{\mathbb{C}}) - \mu_{\mathbb{R}}(\varphi^{\frac{N}{2}}) \leq 2 + \frac{2g-2s}{|\varphi|-1} \leq 2(g+1-s)$. If $|\varphi| \geq g+2-s$ then $\mu(\varphi_{\mathbb{C}}) - \mu_{\mathbb{R}}(\varphi^{\frac{N}{2}}) \leq 2$.*

Proof: By Lemma 2.3, N is even, the real ramification points of $\pi : X \rightarrow X/\langle\varphi\rangle$ are the real fixed points of $\varphi^{\frac{N}{2}}$ and $\mu_{\mathbb{R}}(\varphi^i) = 0$ for any $i \in \{1, \dots, N-1\}$ such that $i \neq \frac{N}{2}$. We denote by X' the quotient space $X/\langle\varphi\rangle$ and by g' the genus of X' .

By Lemma 2.3, there are exactly $\frac{\mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})}{2}$ connected components of $X(\mathbb{R})$ containing at least a real ramification point of π . More precisely, each of these connected components contains exactly two real ramification points of π (see Lemma 2.3). It means that there are $\frac{\mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})}{2}$ connected components of $X(\mathbb{R})$ such that the image by π of each of these components is not a connected component of $X'(\mathbb{R})$. Consequently, there are $s - \frac{\mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})}{2}$ connected components of $X(\mathbb{R})$ such that the image by π of each of these components is a connected component of $X'(\mathbb{R})$. Let v denote the number of connected components of $X'(\mathbb{R})$ that do not contained any real branch point of π . By a previous computation, we have $v \geq \frac{s - \frac{\mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})}{2}}{|\varphi|}$. Since π has at least 2 real branch points, from Harnack inequality, we get

$$g' \geq v \geq \frac{s - \frac{\mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})}{2}}{|\varphi|}$$

i.e.

$$s - \frac{\mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})}{2} \leq g'|\varphi|. \quad (4)$$

From (3), we obtain $\mu(\varphi_{\mathbb{C}}) \leq 2 + \frac{2g-2|\varphi|g'}{|\varphi|-1}$. Combining the previous inequality and (4), we get

$$\mu(\varphi_{\mathbb{C}}) \leq 2 + \frac{2g - 2s + \mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})}{|\varphi| - 1}.$$

The rest of the proof follows easily from the previous inequality. \square

2.2 Non-real fixed points of automorphisms of real curves

The following theorem gives an upper bound on the number of non-real fixed points of an automorphism in terms of the number of connected component of the real part of the curve.

Theorem 2.9 *Let φ be a non-trivial automorphism of X such that $\pi : X \rightarrow X' = X/\langle\varphi\rangle$ is without real ramification points. Then*

$$\mu(\varphi_{\mathbb{C}}) \leq 4 + 2\frac{g+1-s}{|\varphi|-1} \text{ if } s > 1$$

and

$$\mu(\varphi_{\mathbb{C}}) \leq 4 + 2 \frac{g-1}{|\varphi|-1} \text{ if } s = 1.$$

Consequently, $\mu(\varphi_{\mathbb{C}}) \leq 4$ if $|\varphi| \geq g+3-s$ (resp. $|\varphi| \geq g+1$) and if $s > 1$ (resp. $s = 1$).

Proof: Let s' denote the number of connected components of $X'(\mathbb{R})$. Since $\pi : X \rightarrow X' = X/\langle \varphi \rangle$ is without real ramification points, the image of any connected component of $X(\mathbb{R})$ is a connected component of $X'(\mathbb{R})$ (Proposition 2.1 (iv)) i.e. $\text{Int}(\pi) = 0$. Let g' denote the genus of X' . By Harnack inequality,

$$s \leq |\varphi|s' \leq |\varphi|(g'+1). \quad (5)$$

From (3) and (5), we obtain respectively $\mu(\varphi_{\mathbb{C}}) \leq 2 + \frac{2g-2|\varphi|g'}{|\varphi|-1}$ and $|\varphi|g' \geq s - |\varphi|$ (since $|\varphi| \geq 2$ we replace the last inequality by $|\varphi|g' \geq 2 - |\varphi|$ in the case $s = 1$). Combining the two previous inequalities, we get

$$\mu(\varphi_{\mathbb{C}}) \leq 4 + 2 \frac{g+1-s}{|\varphi|-1} \quad (6)$$

if $s > 1$, and

$$\mu(\varphi_{\mathbb{C}}) \leq 4 + 2 \frac{g-1}{|\varphi|-1} \quad (7)$$

if $s = 1$. Since φ is real, $\mu(\varphi_{\mathbb{C}})$ is even. By (6) (resp.(7)), $\mu(\varphi_{\mathbb{C}}) \leq 4$ if $\frac{g+1-s}{|\varphi|-1} < 1$ and $s > 1$ (resp. $\frac{g-1}{|\varphi|-1} < 1$ and $s = 1$) i.e. if $|\varphi| \geq g+3-s$ (resp. $|\varphi| \geq g+1$). \square

Let us state a nice consequence of the previous theorem for M-curves.

Corollary 2.10 *Let φ be a non-trivial automorphism of an M-curve X such that $\pi : X \rightarrow X' = X/\langle \varphi \rangle$ is without real ramification points. Then $\mu(\varphi_{\mathbb{C}}) \leq 4$.*

For an automorphism φ of order $|\varphi| \geq g+2$ we may improve the result of Theorem 2.9.

Proposition 2.11 *Let φ be an automorphism of a curve X such that $\pi : X \rightarrow X' = X/\langle \varphi \rangle$ is without real ramification points. If $|\varphi| \geq g+2$ then $\mu(\varphi_{\mathbb{C}}) \leq 2$.*

Proof: From (3), we obtain $\mu(\varphi_{\mathbb{C}}) \leq 2 + \frac{2g-2|\varphi|g'}{|\varphi|-1} \leq 2 + \frac{2g}{|\varphi|-1}$. If $|\varphi| \geq g+2$ then we get $\mu(\varphi_{\mathbb{C}}) \leq 2$. \square

3 An upper bound on the order of some automorphisms groups of real curves with real ramification points

We give an upper bound for the order of an automorphism φ such that $\pi : X \rightarrow X/\langle \varphi \rangle$ has at least one real ramification point.

3.1 The cyclic case

Theorem 3.1 *Let φ be an automorphism of X of order $N > 1$ such that $\pi : X \rightarrow X/\langle\varphi\rangle$ has at least one real ramification point. Then*

$$N = \frac{\mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})}{\text{Int}(\pi)} \leq \inf \left\{ \frac{2s}{\text{Int}(\pi)}, \frac{2g+2-4g''}{\text{Int}(\pi)} \right\} \leq 2g+2$$

where g'' denotes the genus of $X/\langle\varphi^{\frac{N}{2}}\rangle$. The decomposition of $\sigma(\varphi)$ contains $\text{Int}(\pi)$ disjoint cycles of order $\frac{N}{2}$. If a connected component C of $X(\mathbb{R})$ contains a real ramification point of π then $\text{Stab}(C) = \mathbb{Z}/2\mathbb{Z}$.

Proof: By Lemma 2.3, the real ramification points of $\pi : X \rightarrow X/\langle\varphi\rangle$ are the real fixed points of $\varphi^{\frac{N}{2}}$ and $\mu_{\mathbb{R}}(\varphi^i) = 0$ for any $i \in \{1, \dots, N-1\}$ such that $i \neq \frac{N}{2}$. Let r denote the number of real ramification points of π . According to the above remarks, we have $r = \mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})$. The real ramification points of π are contained in fibers above $2\text{Int}(\pi)$ real branch points of $X/\langle\varphi\rangle$ which correspond to end-points of the image by π of some connected components of $X(\mathbb{R})$ (see Proposition 2.1). In each fiber above one of these real branch points of $X/\langle\varphi\rangle$, there are $\frac{N}{2}$ real ramification points of π contained in distinct connected components of $X(\mathbb{R})$ (see Lemma 2.3) and it is easy to check that φ operates on these $\frac{N}{2}$ real ramification points as a cycle of order $\frac{N}{2}$. Hence

$$r = 2\text{Int}(\pi)\frac{N}{2},$$

and the decomposition of $\sigma(\varphi)$ contains $\text{Int}(\pi)$ disjoint cycles of order $\frac{N}{2}$. By Proposition 2.3, $r \leq 2s$ hence $N \leq \frac{2s}{\text{Int}(\pi)} \leq 2s$ since $\text{Int}(\pi) \geq 1$. By (3), $\mu(\varphi_{\mathbb{C}}^{\frac{N}{2}}) \leq 2g+2-4g''$. Hence $N = \frac{r}{\text{Int}(\pi)} \leq \frac{2g+2-4g''}{\text{Int}(\pi)}$. By Proposition 2.4, if a connected component C of $X(\mathbb{R})$ contains a real ramification point of π then $\text{Stab}(C) = \mathbb{Z}/2\mathbb{Z}$. \square

We will now look at the case of an automorphism of maximum order.

Theorem 3.2 *Let φ be an automorphism of X of order $2g+2$ such that $\pi : X \rightarrow X/\langle\varphi\rangle$ has at least one real ramification point. Then X is an hyperelliptic M-curve of even genus, $X/\langle\varphi\rangle \simeq \mathbb{P}_{\mathbb{R}}^1$, $\text{Int}(\pi) = 1$, $w_{\mathbb{R}}(\pi) = 2$ and the ramification index of the fibers over the 2 real branch points is 2, $w_{\mathbb{C}}(\pi) = 1$ and the ramification index of the fibers over the 2 conjugate non-real branch points of $\pi_{\mathbb{C}}$ is $g+1$. Moreover $\sigma(\varphi)$ is the cyclic permutation $(1\ 2 \dots g+1)$, φ^{g+1} is the hyperelliptic involution and $\text{Stab}(C) = \mathbb{Z}/2\mathbb{Z} = \langle\varphi^{g+1}\rangle$ for any connected component C of $X(\mathbb{R})$.*

Proof: Since $|\varphi| = 2g+2$, Theorem 3.1 implies $s = g+1$ and $X/\langle\varphi^{\frac{|\varphi|}{2}}\rangle \simeq \mathbb{P}_{\mathbb{R}}^1$. Consequently, X is an hyperelliptic M-curve and φ^{g+1} is the hyperelliptic involution. Moreover, $\text{Int}(\pi) = \frac{\mu_{\mathbb{R}}(\varphi^{g+1})}{2g+2} = 1$, $\sigma(\varphi)$ is the cyclic permutation $(1\ 2 \dots g+1)$ and $\text{Stab}(C) = \mathbb{Z}/2\mathbb{Z} = \langle\varphi^{g+1}\rangle$ for any connected component C of $X(\mathbb{R})$, again by Theorem 3.1. Since $X/\langle\varphi^{g+1}\rangle \simeq \mathbb{P}_{\mathbb{R}}^1$ then clearly $X/\langle\varphi\rangle \simeq \mathbb{P}_{\mathbb{R}}^1$.

Assume g is odd and consider the morphism of degree 4, $\pi''' : X \rightarrow X/\langle\varphi^{\frac{g+1}{2}}\rangle = X'''$. Let g''' denote the genus of X''' . By Theorem 2.6 and Lemma 2.3, $\varphi_{\mathbb{C}}^{\frac{g+1}{2}}$ is fixed point free and $\mu(\varphi_{\mathbb{C}}^{g+1}) - \mu_{\mathbb{R}}(\varphi^{g+1}) = 0$. Consequently, the ramification points of π''' are the $2g + 2$ real fixed points of φ^{g+1} . According to Theorem 3.1 and Proposition 2.5, we have $w_{\mathbb{R}}(\pi''') = 2\text{Int}(\pi''') = g + 1$. Writting (3) for $\varphi^{\frac{g+1}{2}}$, we have $2g - 2 = 4(2g''' - 2) + 2g + 2$ i.e. $1 = 2g'''$, which is impossible. Hence g is even.

Using the results of [Kr-Ne] and since $X/\langle\varphi\rangle \simeq \mathbb{P}_{\mathbb{R}}^1$, we see that π does not have any real branch point such that the fiber over this point is non-real. Consequently $w_{\mathbb{R}}(\pi) = 2\text{Int}(\pi) = 2$. Let $Q_1, \dots, Q_{w_{\mathbb{C}}(\pi)}$ denote the non-real branch points of π . The Riemann-Hurwitz relation (2) gives

$$2g - 2 = (2g + 2)(-2) + 2g + 2 + 2(2g + 2) \sum_{i=1}^{w_{\mathbb{C}}(\pi)} \left(1 - \frac{1}{e(Q_i)}\right).$$

Hence

$$\sum_{i=1}^{w_{\mathbb{C}}(\pi)} \left(1 - \frac{1}{e(Q_i)}\right) = \frac{g}{g+1} < 1. \quad (8)$$

If $e(Q_i) = 2$ for a $i \in \{1, \dots, w_{\mathbb{C}}(\pi)\}$ then, writting $Q_i = Q'_i + \bar{Q}'_i$ on $X_{\mathbb{C}}$, the points Q'_i, \bar{Q}'_i are fixed points of $\varphi_{\mathbb{C}}^{g+1}$ in contradiction with Theorem 2.6. Hence $e(Q_i) > 2$ for $i = 1, \dots, w_{\mathbb{C}}(\pi)$. By (8) we conclude that $w_{\mathbb{C}}(\pi) = 1$, hence that $e(Q_1) = g + 1$. Since, if g is odd, a fixed point of $\varphi_{\mathbb{C}}^2$ is also a fixed point of $\varphi_{\mathbb{C}}^{g+1}$, it also follows from Theorem 2.6 that g is even. \square

We give now some remarks concerning the genus of $X/\langle\varphi^{\frac{|\varphi|}{2}}\rangle$.

Proposition 3.3 *Let φ be an automorphism of X of order $N > 1$ such that $\pi : X \rightarrow X/\langle\varphi\rangle$ has at least one real ramification point. Let g'' denote the genus of $X/\langle\varphi^{\frac{N}{2}}\rangle$. If $X/\langle\varphi\rangle \simeq \mathbb{P}_{\mathbb{R}}^1$ then $g'' \leq \frac{g+1-s}{2}$.*

Proof: Since $X/\langle\varphi\rangle \simeq \mathbb{P}_{\mathbb{R}}^1$ and since π has at least one real ramification point, it follows from Proposition 2.1 that the image by π of any connected component of $X(\mathbb{R})$ is strictly contained in $\mathbb{P}_{\mathbb{R}}^1(\mathbb{R})$. By Lemma 2.3, any connected component of $X(\mathbb{R})$ contain exactly 2 real ramification points of π . Since the real ramification points of $\pi : X \rightarrow X/\langle\varphi\rangle$ are the real fixed points of $\varphi^{\frac{N}{2}}$ (Lemma 2.3), we conclude that $2s \leq \mu(\varphi_{\mathbb{C}}^{\frac{N}{2}}) = 2g + 2 - 4g''$, which proves the proposition. \square

For a real curve X with many real components, the previous proposition yields information about the automorphisms φ of X such that $X/\langle\varphi\rangle \simeq \mathbb{P}_{\mathbb{R}}^1$.

Corollary 3.4 *Let φ be an automorphism of X of order N such that $\pi : X \rightarrow X/\langle\varphi\rangle$ has at least one real ramification point. If $X/\langle\varphi\rangle \simeq \mathbb{P}_{\mathbb{R}}^1$ and X has many real components then X is hyperelliptic.*

For an automorphism of prime order, the existence of a real ramification point for the quotient map forces this automorphism to be an involution.

Proposition 3.5 *Let φ be an automorphism of X of prime order p such that $\pi : X \rightarrow X/\langle\varphi\rangle$ has at least one real ramification point. Then $p = 2$.*

Proof: If $\pi : X \rightarrow X/\langle\varphi\rangle$ has at least one real ramification point then $\langle\varphi\rangle$ has a subgroup of order 2. Hence $p = 2$. \square

3.2 The abelian case

We give an upper bound for the order of an abelian group of automorphisms G such that $\pi : X \rightarrow X/G$ has at least one real ramification point.

Theorem 3.6 *Let G be an abelian group of automorphisms of X of order $N > 1$ such that $\pi : X \rightarrow X/G$ has at least one real ramification point. Then*

$$|G| \leq \inf \left\{ \frac{4s}{\text{Int}(\pi)}, 2g + 2 + 4(g + 1 - s) \right\} \leq 3g + 3.$$

Proof: Let g' denote the genus of X' . The real ramification points of $\pi : X \rightarrow X/G$ are contained in fibers above $2\text{Int}(\pi)$ real branch points of X/G which correspond to end-points of the image by π of some connected components of $X(\mathbb{R})$ (see Proposition 2.1). Let P be a real ramification point of π . By [Pa, Lem. 1.1], the stabilizer subgroup of P in G is cyclic. Let φ be a generator of $\text{Stab}(P)$, then P is a real fixed point of φ and moreover we have $|\varphi| = 2$ (see Proposition 2.1). In the fiber above $Q = \pi(P)$, we have $\frac{N}{2}$ real fixed points of φ since points in the same fiber have conjugate stabilizers and since G is abelian. By Lemma 2.3, a connected component of $X(\mathbb{R})$ contains at most 2 real fixed points of φ . Hence $\pi^{-1}(Q)$ intersects at least $\frac{N}{4}$ distinct connected components of $X(\mathbb{R})$ i.e. $\frac{N}{4} \leq s$. The same conclusion can be drawn for any ramified fiber with real points, which proves that $N \leq \frac{4s}{\text{Int}(\pi)}$.

Before finishing the proof we need to make a remark concerning real branch points with non-real fiber in the case $g' = 0$. Let Q be a real branch point such that $\pi^{-1}(Q)$ is non-real. By [Kr-Ne, Satz 1], the decomposition group of Q in G is the dihedral group $D_{e(Q)}$. Since G is abelian, we must have $e(Q) = 2$.

Now assume $N > 2g + 2$. It follows from the beginning of the proof that $\text{Int}(\pi) = 1$. By Proposition 2.1 (iv), there exist exactly two real branch points Q_1, Q_2 of π with real fibers and with $e(Q_1) = e(Q_2) = 2$. Let P_1 (resp. P_2) be a point in the fiber $\pi_{\mathbb{C}}^{-1}(Q_1)$ (resp. $\pi_{\mathbb{C}}^{-1}(Q_2)$). Let φ_1 (resp. φ_2) be a generator of $\text{Stab}(P_1)$ (resp. $\text{Stab}(P_2)$). Since G is abelian, all the points in the fiber $\pi_{\mathbb{C}}^{-1}(Q_1)$ (resp. $\pi_{\mathbb{C}}^{-1}(Q_2)$) are fixed points of φ_1 (resp. φ_2). Since $N > 2g + 2$, it follows from the Riemann-Hurwitz formula (2) that $g' = 0$ and there exist at least two distinct branch points Q'_1, Q'_2 of $\pi_{\mathbb{C}}$ with non-real fibers. We may assume that the points Q'_1, Q'_2 are either both real or both non-real and conjugate. Let φ'_1 (resp. φ'_2) be a generator of the stabilizer subgroup of any point in $\pi_{\mathbb{C}}^{-1}(Q'_1)$ (resp. $\pi_{\mathbb{C}}^{-1}(Q'_2)$). We have different cases.

Case 1: φ_1 and φ_2 are powers of either of φ'_1 or φ'_2 .

By Theorem 2.8 and since there are $\frac{|G|}{e(Q'_1)}$ points in the fiber $\pi_{\mathbb{C}}^{-1}(Q'_1)$, we obtain $\frac{|G|}{e(Q'_1)} \leq 2(g + 1 - s)$ and X is not an M-curve. Similarly, we have $\frac{|G|}{e(Q'_2)} \leq 2(g + 1 - s)$. By

(2) and according to above remarks, we obtain $2g - 2 \geq -2|G| + \sum_{i=1}^2 |G|(1 - \frac{1}{e(Q_i)}) + \sum_{i=1}^2 (|G| - \frac{|G|}{e(Q_i)}) \geq |G| - 4(g + 1 - s)$. Hence $|G| \leq 2g - 2 + 4(g + 1 - s)$.

Case 2: φ_1 is a power of φ'_1 , and φ_2 is not a power of either of φ'_1 and φ'_2 . By Theorem 2.8, we have $\frac{|G|}{e(Q_1)} \leq \mu(\varphi'_{1,\mathbb{C}}) - \mu_{\mathbb{R}}(\varphi_1) \leq 2(g + 1 - s)$ and X is not an M-curve. Since φ_2 is not a power of either of φ'_1 and φ'_2 and since the fibers $\pi_{\mathbb{C}}^{-1}(Q_1)$ and $\pi_{\mathbb{C}}^{-1}(Q_2)$ contain all the real fixed points of the elements of G , it follows that the map $X \rightarrow X/\langle \varphi'_2 \rangle$ has no real ramification point. By (6), we have $\frac{|G|}{e(Q_2)} \leq \mu(\varphi'_{2,\mathbb{C}}) \leq 4 + 2(g + 1 - s)$. By (2) and according to the above remarks, we obtain $2g - 2 \geq -2|G| + \sum_{i=1}^2 |G|(1 - \frac{1}{e(Q_i)}) + \sum_{i=1}^2 (|G| - \frac{|G|}{e(Q_i)}) \geq |G| - 4 - 4(g + 1 - s)$. Hence $|G| \leq 2g + 2 + 4(g + 1 - s)$.

Case 3: φ_1 and φ_2 are not powers of either of φ'_1 or φ'_2 and $e(Q'_1) \geq 3$. Similarly to the previous case, the maps $X \rightarrow X/\langle \varphi'_1 \rangle$ and $X \rightarrow X/\langle \varphi'_2 \rangle$ are without real ramification points. By an above remark, Q'_1 is non-real and we may assume that Q'_1 and Q'_2 are two conjugate points of $X'_{\mathbb{C}}$ corresponding to the non-real point Q' of X' . By (6) and since $e(Q') \geq 3$, we have $\frac{|G|}{e(Q')} \leq 2 + \frac{g+1-s}{2}$. By (2) and according to the above remarks, we obtain $2g - 2 \geq -2|G| + \sum_{i=1}^2 |G|(1 - \frac{1}{e(Q_i)}) + 2(|G| - \frac{|G|}{e(Q')}) \geq |G| - 4 - (g + 1 - s)$. Hence $|G| \leq 2g + 2 + (g + 1 - s)$. Since we have assumed $N > 2g + 2$, this case does not occur when X is an M-curve.

Case 4: φ_1 and φ_2 are not powers of either of φ'_1 or φ'_2 and $e(Q'_1) = e(Q'_2) = 2$. By (2), there exists a third branch point Q'_3 of $\pi_{\mathbb{C}}$ with non-real fibers. By Theorem 2.8 and (6), we obtain $\frac{|G|}{e(Q'_3)} \leq 4 + 2(g + 1 - s)$. By (2), we get $2g - 2 \geq -2|G| + \sum_{i=1}^2 |G|(1 - \frac{1}{e(Q_i)}) + \sum_{i=1}^3 (|G| - \frac{|G|}{e(Q'_i)}) \geq |G| - 4 - 2(g + 1 - s)$. Hence $|G| \leq 2g + 2 + 2(g + 1 - s)$. Since we have assumed $N > 2g + 2$, this case does not occur when X is an M-curve.

We have proved that $|G| \leq \inf\{4s, 2g + 2 + 4(g + 1 - s)\}$. Since $4s \geq 2g + 2 + 4(g + 1 - s)$ if and only if $s \geq \frac{3}{4}(g + 1)$, we get $|G| \leq 3g + 3$. \square

For M-curves, we obtain:

Corollary 3.7 *Let G be an abelian group of automorphisms of an M-curve X such that $\pi : X \rightarrow X/G$ has at least one real ramification point. Then $|G| \leq 2g + 2$.*

3.3 The hyperelliptic case

We will give an upper bound for the order of the group of automorphisms of a real hyperelliptic curve such that the hyperelliptic involution has at least one real fixed point. Before that, we will prove a more general result.

Theorem 3.8 *Let G be a group of automorphism of X of order $N > 1$ such that there exists $\varphi \neq \text{Id}$ in the center $Z(G)$ of G with at least one real fixed point. Then $|G| \leq \inf\{4s, 4g + 4 - 8g''\} \leq 4g + 4$ where g'' denotes the genus of $X/\langle \varphi \rangle$.*

Proof: Let P be a real fixed point of φ , then P is a real ramification point of $\pi : X \rightarrow X/G$ with ramification index $e_P = 2$ (see Proposition 2.1). Since the points in

the fiber above $Q = \pi(P)$ have conjugate stabilizer subgroups and since $\varphi \in Z(G)$, the fiber $\pi^{-1}(Q)$ is composed by $\frac{N}{2}$ real points which are real fixed points of φ . By Lemma 2.3, a connected component of $X(\mathbb{R})$ contains at most 2 real fixed points of φ . Hence $\pi^{-1}(Q)$ intersects at least $\frac{N}{4}$ distinct connected components of $X(\mathbb{R})$ i.e. $\frac{N}{4} \leq s$. By (3), $\mu_{\mathbb{R}}(\varphi) \leq 2g + 2 - 4g''$. Hence $\frac{N}{2} \leq 2g + 2 - 4g''$ and the proof is done. \square

Corollary 3.9 *Let X be a real hyperelliptic curve such that the hyperelliptic involution ι has at least a real fixed point (e.g. if $s \geq 3$). Then $|\text{Aut}(X)| \leq 4s$.*

Proof: By [Fa-K, Cor. 3 p. 102], the hyperelliptic involution ι is in the center of $\text{Aut}(X)$. If $s \geq 3$ then ι has at least a real fixed point [Mo, Prop. 4.3]. The rest of the proof follows from Theorem 3.8. \square

Remark 3.10 The order of the automorphism group of a real curve X cannot be larger than $12(g-1)$ [Ma1]. In the case $|\text{Aut}(X)| = 12(g-1)$, the map $\pi : X \rightarrow X/\text{Aut}(X)$ has at least a real ramification point [Ma1] and it follows from Corollary 3.9 that X is not hyperelliptic. In the cyclic case, the curves with an automorphism of maximum order are hyperelliptic.

We extend the result concerning the hyperelliptic curves to real curves which are 2-sheeted coverings.

Corollary 3.11 *Let X be a real curve such that X is a 2-sheeted covering of a curve of genus g'' . Assume $g > 4g'' + 1$ and the involution induced by the 2-sheeted covering has at least a real fixed point, then $|\text{Aut}(X)| \leq \inf\{4s, 4g + 4 - 8g''\} \leq 4g + 4$.*

Proof: If $g > 4g'' + 1$, the involution induced by the 2-sheeted covering is in the center of $\text{Aut}(X)$ [Fa-K, Thm. p. 250]. The proof follows now from Theorem 3.8. \square

4 An upper bound for some groups of automorphisms of real curves without real ramification points

This section is devoted to the study of automorphisms of real curves without real fixed points.

4.1 The cyclic case for M-curves

In this section we give an upper bound on the order of an automorphism of an M-curve using Corollary 2.10.

Before giving this bound, we need to give a result concerning the number of branch points of a map corresponding to a quotient by an abelian group of automorphisms.

Lemma 4.1 *Let G be an abelian group of automorphisms of a smooth projective curve X over \mathbb{C} . Let π denote the map $\pi : X \rightarrow X' = X/G$. Then $w(\pi) \neq 1$.*

Proof: Write G as the direct sum of the cyclic subgroups G_1, \dots, G_t generated by $\varphi_1, \dots, \varphi_t$. Assume π has a unique branch point denoted by Q . From the algebra structure of $\pi_* \mathcal{O}_X$ and from the action of G on it, it is possible to derive a linear equivalence, for $i = 1, \dots, t$, between $|\varphi_i|D_i$ and $n_i Q$, with D_i a divisor on X' associated to the dual of $\langle \varphi_i \rangle$ and n_i an integer such that $0 \leq n_i < |\varphi_i|$ [Pa, Prop. 2.1]. By [Pa, Prop. 2.1], the data: D_i , Q and the linear equivalence between $|\varphi_i|D_i$ and $n_i Q$, determines uniquely π up to isomorphism. According to the previous remark and since π is ramified, we can assume that $n_1 \geq 1$ (see [Pa, Example 2.1 (ii)]). But then the condition $n_1 < |\varphi_1|$ says that the linear equivalence between $|\varphi_1|D_1$ and $n_1 Q$ is impossible. Hence $w(\pi) > 1$. \square

In the following theorem, we determine all the automorphisms of M-curves.

Theorem 4.2 *Let φ be a non-trivial automorphism of an M-curve X such that $\pi : X \rightarrow X' = X/\langle \varphi \rangle$ is without real ramification points. Let g' denote the genus of X' . Then one of the five possibilities occurs.*

- (i) $\mu(\varphi_{\mathbb{C}}) = 4$, $|\varphi| = \frac{g+1}{g'+1}$, X' is an M-curve, $\pi_{\mathbb{C}}$ has 4 branch points $Q_1, \bar{Q}_1, Q_2, \bar{Q}_2$ i.e. $w_{\mathbb{C}}(\pi) = 2$ and $w_{\mathbb{R}}(\pi) = 0$, $e(Q_1) = e(Q_2) = |\varphi|$.
- (ii) $\mu(\varphi_{\mathbb{C}}) = 2$, $|\varphi| = \frac{g}{g'}$ with $g' \geq 1$, X' is an M-curve, $\pi_{\mathbb{C}}$ has 2 branch points Q_1, \bar{Q}_1 i.e. $w_{\mathbb{C}}(\pi) = 1$ and $w_{\mathbb{R}}(\pi) = 0$, $e(Q_1) = |\varphi|$.
- (iii) $\mu(\varphi_{\mathbb{C}}) = 0$, $|\varphi| = \frac{g+1}{g'}$ with $g' \geq 1$, X' has many real components, $\pi_{\mathbb{C}}$ has 2 branch points Q_1, \bar{Q}_1 i.e. $w_{\mathbb{C}}(\pi) = 1$ and $w_{\mathbb{R}}(\pi) = 0$, $e(Q_1) = \frac{|\varphi|}{2}$.
- (iv) $\mu(\varphi_{\mathbb{C}}) = 0$, $|\varphi| = \frac{g+1}{g'}$ with $g' \geq 1$, X' is an M-curve, $\pi_{\mathbb{C}}$ has 2 real branch points Q_1, Q_2 with non-real fibers i.e. $w_{\mathbb{C}}(\pi) = 0$ and $w_{\mathbb{R}}(\pi) = 2$, $e(Q_1) = e(Q_2) = \frac{|\varphi|}{2}$. The real branch points Q_1, Q_2 are contained in the same connected component of $X'(\mathbb{R})$.
- (v) $\mu(\varphi_{\mathbb{C}}) = 0$, $|\varphi| = \frac{g-1}{g'-1}$ with $g' \geq 2$, X' has many real components, $\pi_{\mathbb{C}}$ does not have any branch point.

It follows that $|\varphi| \leq g+1$ and X' has many real components.

Proof: Let s' denote the number of connected components of $X'(\mathbb{R})$. By Corollary 2.10, $\mu(\varphi_{\mathbb{C}}) \leq 4$. We will proceed by looking successively at the cases $\mu(\varphi_{\mathbb{C}}) = 4, 2$ and 0.

Assume $\mu(\varphi_{\mathbb{C}}) = 4$. By Corollary 2.10 and since a fixed point of $\varphi_{\mathbb{C}}$ is fixed for any power of $\varphi_{\mathbb{C}}$, the Riemann-Hurwitz relation (3) reads $2g - 2 = |\varphi|(2g' - 2) + 4(|\varphi| - 1)$. It gives $|\varphi| = \frac{g+1}{g'+1}$. By (5) we have $s' \geq \frac{g+1}{|\varphi|} = g' + 1$. Harnack inequality says that X' is an M-curve. We are in the case of statement (i).

Assume $\mu(\varphi_{\mathbb{C}}) = 2$. Since a totally ramified non-real point of π is necessarily in a fiber above a non-real branch point, $\pi_{\mathbb{C}}$ has 2 branch points Q_1, \bar{Q}_1 such that $e(Q_1) = |\varphi|$. Let

P_1 (resp. \bar{P}_1) denote the totally ramified point over Q_1 (resp. \bar{Q}_1). If $\pi_{\mathbb{C}}$ does not have more branch points, then (3) gives $|\varphi| = \frac{g}{g'}$ with $g' \geq 1$. By (5), X' is an M-curve and we get statement (ii). If $\pi_{\mathbb{C}}$ has at least one more branch point then set

$$j = \inf\{i > 1, \varphi_{\mathbb{C}}^i \text{ has a fixed point } P_2 \text{ and } P_1 \text{ is not fixed by } \varphi_{\mathbb{C}}^i\}.$$

Since the fiber containing P_2 contains j fixed points of $\varphi_{\mathbb{C}}^j$ and since P_1 and \bar{P}_1 are fixed points of $\varphi_{\mathbb{C}}^j$, Corollary 2.10 implies that $j = 2$ and that P_2 and \bar{P}_2 are contained in a fiber above a real branch point. It follows that $w(\pi_{\mathbb{C}}) = 3$. By (2), we get $2g - 2 = |\varphi|(2g' - 2) + 2(|\varphi| - 1) + |\varphi| - 2$ i.e. $|\varphi| = \frac{2g+2}{2g'+1}$. Since π has a real branch point and since $\pi : X \rightarrow X' = X/\langle\varphi\rangle$ is without real ramification points i.e. $\text{Int}(\pi) = 0$, the real branch point is contained in a connected component C' of $X'(\mathbb{R})$ such that $\pi^{-1}(C')$ is totally non real. Hence $s' \geq 2$ and $g' \geq 1$ by Harnack inequality. So we may refine (5) in this case and we get $g + 1 \leq |\varphi|g' = g' \frac{2g+2}{2g'+1}$. It gives a contradiction and the single case of an automorphism φ with $\mu(\varphi_{\mathbb{C}}) = 2$ is given by statement (ii).

Finally, assume $\varphi_{\mathbb{C}}$ is fixed point free and $\pi_{\mathbb{C}}$ does not have any branch point. From (2) and (5), it follows that $|\varphi| = \frac{g-1}{g'-1}$ with $g' \geq 2$ and that X' has many real components. Assume $\varphi_{\mathbb{C}}$ is fixed point free and $w(\pi_{\mathbb{C}}) \geq 1$. Let

$$j = \inf\{i > 1, \varphi_{\mathbb{C}}^i \text{ has a fixed point}\}.$$

Let P_1 be a fixed point of $\varphi_{\mathbb{C}}^j$. Then \bar{P}_1 is also a fixed point of $\varphi_{\mathbb{C}}^j$. Firstly, assume that P_1 and \bar{P}_1 are contained in two conjugate fibers above two conjugate branch points denoted by Q_1 and \bar{Q}_1 . Since the fiber containing P_1 contains j fixed points of $\varphi_{\mathbb{C}}^j$, Corollary 2.10 implies that $j = 2$. By Corollary 2.10, we see that a ramified fiber of $\pi_{\mathbb{C}}$ contains at most 4 points and the number of points in the fiber is exactly the smallest power of φ which generates the stabilizer group of any point in the fiber. Since a non-real fiber over a real point and two conjugate fibers contain both an even number of points, we conclude that a ramified fiber contains 2 or 4 points. Hence $w(\pi_{\mathbb{C}}) = 2$ since if we assumed $w(\pi_{\mathbb{C}}) > 2$, we would have $\mu(\varphi_{\mathbb{C}}^2) > 4$ or $\mu(\varphi_{\mathbb{C}}^4) > 4$ which contradicts Corollary 2.10 (a fixed point of $\varphi_{\mathbb{C}}^2$ is also a fixed point of $\varphi_{\mathbb{C}}^4$). By (2), we get $2g - 2 = |\varphi|(2g' - 2) + 2(|\varphi| - 2)$ i.e. $|\varphi| = \frac{g+1}{g'}$ and $g' \geq 1$. By (5), X' has many real components and we get statement (iii). Assume now that P_1 and \bar{P}_1 are contained in the same fiber above a real branch point denoted by Q_1 . By Lemma 4.1, $\pi_{\mathbb{C}}$ has at least one more branch point. From Corollary 2.10 and according to the above remarks, it follows that $\pi_{\mathbb{C}}$ has exactly 2 ramified fibers over two real branch points Q_1 and Q_2 , one fiber contains P_1 and \bar{P}_1 and the other contains the points P_2 and \bar{P}_2 . Moreover, $e(Q_1) = e(Q_2) = \frac{|\varphi|}{2}$. From (2), we get $|\varphi| = \frac{g+1}{g'}$ and $g' \geq 1$. Arguing as in the proof of the case $\mu(\varphi_{\mathbb{C}}) = 2$, we see that the inverse image by π of the connected components of $X'(\mathbb{R})$ containing Q_1 and Q_2 are totally non-real. We may refine (5) in this case and we get $g + 1 \leq |\varphi|(s' - 1) \leq |\varphi|g' = g + 1$. Hence X' is an M-curve and Q_1 and Q_2 are contained in the same connected component of $X'(\mathbb{R})$. We are in the case of statement (iv) and the proof is done. \square

4.2 The cyclic case

At the beginning of this section we give an upper bound for the order of an automorphism of a real curve and we study the limit cases.

Theorem 4.3 *Let φ be a non-trivial automorphism of a real curve X such that $\pi : X \rightarrow X' = X/\langle \varphi \rangle$ is without real ramification points. Then $|\varphi| \leq \sup\{2g + 4 - s, 2g + 2 - \frac{2}{3}s\}$ if $s > 1$ and $|\varphi| \leq 2g + 2$ if $s = 1$.*

Proof: Let g' denote the genus of X' . We assume that $|\varphi| \geq g + 3 - s$ and by Theorem 2.9 we have $\mu(\varphi_{\mathbb{C}}) \leq 4$. We will proceed by looking successively at the cases $\mu(\varphi_{\mathbb{C}}) = 4, 2$ and 0. Let s' denote the number of connected components of $X'(\mathbb{R})$.

Case 1: $\mu(\varphi_{\mathbb{C}}) = 4$. The Riemann-Hurwitz relation (3) gives $2g - 2 \geq |\varphi|(2g' - 2) + 4(|\varphi| - 1)$. It gives $|\varphi| \leq \frac{g+1}{g'+1} \leq g + 1$.

Case 2: $\mu(\varphi_{\mathbb{C}}) = 2$. By (3), we get $|\varphi| \leq \frac{g}{g'} \leq g$ if $g' \geq 1$. So assume $g' = 0$. Since $\mu(\varphi_{\mathbb{C}}) = 2$, $\pi_{\mathbb{C}}$ has already 2 branch points Q_1, \bar{Q}_1 (i.e. $w_{\mathbb{C}}(\pi) \geq 1$) such that $e(Q_1) = |\varphi|$ (since $s' = 1$, the two branch points cannot be real). Let P_1 (resp. \bar{P}_1) denote the totally ramified point over Q_1 (resp. \bar{Q}_1). From (3), we see that $\pi_{\mathbb{C}}$ has at least one more branch point. Since $\pi : X \rightarrow X' = X/\langle \varphi \rangle$ is without real ramification points and since $s' = 1$, then $w_{\mathbb{C}}(\pi) \geq 2$. Let Q_2, \bar{Q}_2 denote two conjugate branch points of $\pi_{\mathbb{C}}$ distinct from Q_1 and \bar{Q}_1 . Let $\frac{|\varphi|}{j} = e(Q_2)$. We have $2 \leq j \leq \frac{|\varphi|}{2}$. The Riemann-Hurwitz relation (2) gives $2g - 2 \geq -2|\varphi| + 2|\varphi|(1 - \frac{1}{|\varphi|}) + 2|\varphi|(1 - \frac{j}{|\varphi|})$ i.e.

$$|\varphi| \leq g + j. \quad (9)$$

The stabilizer subgroups of the points in the fibers $\pi_{\mathbb{C}}^{-1}(Q_2)$ and $\pi_{\mathbb{C}}^{-1}(\bar{Q}_2)$ are generated by $\varphi_{\mathbb{C}}^j$ and each fiber contains j points. Moreover, P_1 and \bar{P}_1 are also fixed points of $\varphi_{\mathbb{C}}^j$ since they are fixed points of $\varphi_{\mathbb{C}}$. Consequently, we get $2j + 2 \leq \mu(\varphi_{\mathbb{C}}^j)$. If $s > 1$, using (6) we obtain $2j + 2 \leq 4 + 2\frac{g+1-s}{\frac{|\varphi|}{j}-1}$. Since $2 \leq \frac{|\varphi|}{j}$, we get $j \leq 1 + (g + 1 - s)$. By (9), we get $|\varphi| \leq 2g + 2 - s$ in the case $s > 1$. If $s = 1$, using (7) we obtain $2j + 2 \leq 4 + 2\frac{g-1}{\frac{|\varphi|}{j}-1}$. Similarly to the case $s > 1$, we get $|\varphi| \leq 2g$ in the case $s = 1$.

Case 3: $\mu(\varphi_{\mathbb{C}}) = 0$. Assume $\varphi_{\mathbb{C}}$ is fixed point free and $\pi_{\mathbb{C}}$ does not have any branch point. From (2) and (5), it follows that $|\varphi| = \frac{g-1}{g'-1} \leq g - 1$ with $g' \geq 2$.

In the rest of the proof we assume $\varphi_{\mathbb{C}}$ is fixed point free and that $w(\pi_{\mathbb{C}}) \geq 1$.

Firstly, we assume that $w_{\mathbb{C}}(\pi) \geq 1$ i.e. that $\pi_{\mathbb{C}}$ has two conjugate branch points, denoted by Q_1, \bar{Q}_1 , such that $\frac{|\varphi|}{j} = e(Q_1)$ with $2 \leq j \leq \frac{|\varphi|}{2}$. From (2), we obtain $2g - 2 \geq |\varphi|(2g' - 2) + 2|\varphi|(1 - \frac{j}{|\varphi|})$ i.e.

$$|\varphi|g' \leq g - 1 + j. \quad (10)$$

The $2j$ points in the fibers $\pi_{\mathbb{C}}^{-1}(Q_1)$ and $\pi_{\mathbb{C}}^{-1}(\bar{Q}_1)$ are fixed by $\varphi_{\mathbb{C}}^j$. Consequently, we get $2j \leq \mu(\varphi_{\mathbb{C}}^j)$. If $s > 1$ (resp. $s = 1$), using (6) (resp. (7)) we obtain $2j \leq 4 + 2\frac{g+1-s}{\frac{|\varphi|}{j}-1} \leq 4 + 2(g + 1 - s)$ (resp. $2j \leq 4 + 2\frac{g-1}{\frac{|\varphi|}{j}-1} \leq 4 + 2(g - 1)$). If $g' \geq 1$, it follows from (10) that $|\varphi| \leq 2g + 2 - s$ if $s > 1$ (resp. $|\varphi| \leq 2g$ if $s = 1$).

So assume $g' = 0$. By (2), we see that $\pi_{\mathbb{C}}$ has at least one more branch point. Since $\pi : X \rightarrow X' = X/\langle \varphi \rangle$ is without real ramification points and since $s' = 1$, then $w_{\mathbb{C}}(\pi) \geq 2$. Let Q_2, \bar{Q}_2 denote two conjugate branch points of $\pi_{\mathbb{C}}$ distinct from Q_1 and \bar{Q}_1 . Let $\frac{|\varphi|}{j'} = e(Q_2)$. We have $2 \leq j' \leq \frac{|\varphi|}{2}$.

If $j = \frac{|\varphi|}{2}$ or $j' = \frac{|\varphi|}{2}$, the points in the fibers $\pi_{\mathbb{C}}^{-1}(Q_1)$ and $\pi_{\mathbb{C}}^{-1}(\bar{Q}_1)$, or in the fibers $\pi_{\mathbb{C}}^{-1}(Q_2)$ and $\pi_{\mathbb{C}}^{-1}(\bar{Q}_2)$, are fixed points of $\varphi_{\mathbb{C}}^{\frac{|\varphi|}{2}}$. Hence, using (6) (resp. (7)) we get $|\varphi| \leq 4 + 2(g + 1 - s) = 2g + 6 - 2s \leq 2g + 4 - s$ if $s > 1$ (resp. $|\varphi| \leq 2g + 2$ if $s = 1$).

We assume that $j \leq \frac{|\varphi|}{3}$ and $j' \leq \frac{|\varphi|}{3}$. By (2), we get

$$|\varphi| \leq g - 1 + j + j'. \quad (11)$$

Similarly to the previous cases, we get $2j \leq \mu(\varphi_{\mathbb{C}}^j)$ and $2j' \leq \mu(\varphi_{\mathbb{C}}^{j'})$. Since $3 \leq \frac{|\varphi|}{j}$, it follows from (6) (resp. (7)) that $2j \leq 4 + (g + 1 - s)$ if $s > 1$ (resp. $2j \leq 4 + (g - 1)$ if $s = 1$) and the same is true for j' . Hence $j + j' < 4 + (g + 1 - s)$ if $s > 1$ (resp. $j + j' < g + 3$ if $s = 1$) (equality is impossible in the two previous inequality since it would imply $j = j' = \frac{|\varphi|}{3}$ and we would get a contradiction with (6) and (7)). By (11) and the previous results we get $|\varphi| < 2g + 4 - s$ if $s > 1$ (resp. $|\varphi| < 2g + 2$ if $s = 1$).

Secondly, we turn to the case $w_{\mathbb{C}}(\pi) = 0$ and $w_{\mathbb{R}}(\pi) \geq 1$. From Lemma 4.1 and since $\pi : X \rightarrow X' = X/\langle \varphi \rangle$ is without real ramification points, we have $w_{\mathbb{R}}(\pi) \geq 2$ and $s' \geq 2$. In particular, $g' \geq 1$ since $s' \geq 2$. Let Q_1, Q_2 be two real branch points of π . We set $\frac{|\varphi|}{j} = e(Q_1)$ and $\frac{|\varphi|}{j'} = e(Q_2)$. Since the fibers of π above Q_1 and Q_2 are non-real, we have $2 \leq j \leq \frac{|\varphi|}{2}$ and $2 \leq j' \leq \frac{|\varphi|}{2}$. By (2) and taking account of the previous remarks, we obtain

$$2g'|\varphi| \leq 2g - 2 + j + j'. \quad (12)$$

If $j = \frac{|\varphi|}{2}$ and $j' = \frac{|\varphi|}{2}$, the $j + j'$ points in the fibers $\pi_{\mathbb{C}}^{-1}(Q_1)$ and $\pi_{\mathbb{C}}^{-1}(Q_2)$ are fixed points of $\varphi_{\mathbb{C}}^{\frac{|\varphi|}{2}}$. Hence, using (6) (resp. (7)) we get $j + j' \leq 4 + 2(g + 1 - s)$ if $s > 1$ (resp. $j + j' \leq 2g + 2$ if $s = 1$). Combining (12) and the previous result, we get $|\varphi| \leq 2g + 2 - s$ if $s > 1$ (resp. $|\varphi| \leq 2g$ if $s = 1$).

If $j \leq \frac{|\varphi|}{2}$ (i.e. $e(Q_1) \geq 2$) and $j' < \frac{|\varphi|}{2}$ (i.e. $e(Q_2) \geq 3$). The j' points in the fiber $\pi_{\mathbb{C}}^{-1}(Q_2)$ are fixed points of $\varphi_{\mathbb{C}}^{\frac{|\varphi|}{j'}}$. Using (12), we get

$$|\varphi| \leq \frac{4}{3}g - \frac{4}{3} + \frac{2}{3}j'. \quad (13)$$

Since $e(Q_2) = \frac{|\varphi|}{j'} \geq 3$, using (6) (resp. (7)) and (13), we obtain $|\varphi| \leq 2g + 2 - \frac{2}{3}s$ if $s > 1$ (resp. $|\varphi| \leq 2g + \frac{2}{3}$ if $s = 1$). \square

We will now look at the case of an automorphism of maximum order.

Theorem 4.4 *Let φ be an automorphism of X of order $2g + 2$ such that $\pi : X \rightarrow X/\langle \varphi \rangle$ is without real ramification points. Then X is an hyperelliptic curve of even genus, $s = 1$, $X/\langle \varphi \rangle \simeq \mathbb{P}_{\mathbb{R}}^1$, $w_{\mathbb{R}}(\pi) = 0$ and $w_{\mathbb{C}}(\pi) = 2$. Let $\{Q_1, Q_2, \bar{Q}_1, \bar{Q}_2\}$ be the branch points of $\pi_{\mathbb{C}}$, then $e(Q_1) = 2$, $e(Q_2) = g + 1$. Moreover, φ^{g+1} corresponds to the hyperelliptic involution*

and X is given by the real polynomial equation $y^2 = f(x)$, where f is a monic polynomial of degree $2g + 2$ and where f has no real roots.

Proof: Let g' (resp. g'') denote the genus of $X' = X/\langle\varphi\rangle$ (resp. of $X'' = X/\langle\varphi^{g+1}\rangle$). Looking at the proof of Theorem 4.3, we see that $g' = 0$, $s = 1$ or $s = 2$ and $\pi_{\mathbb{C}}$ has exactly 4 non-real branch points denoted by $Q_1, Q_2, \bar{Q}_1, \bar{Q}_2$. If we set $e(Q_1) = \frac{|\varphi|}{j}$ and $e(Q_2) = \frac{|\varphi|}{j'}$, it follows from the proof of Theorem 4.3 that we can assume $e(Q_1) = 2$ i.e. $j = g + 1$. By (2), we get $j' = 2$ i.e. $e(Q_2) = g + 1$. From the proof of Theorem 4.3, we see that $\mu(\varphi_{\mathbb{C}}^{g+1}) = |\varphi| = 2g + 2$. Hence, considering the relation (3) for φ^{g+1} , we get $g'' = 0$ i.e. X is hyperelliptic and φ^{g+1} is the hyperelliptic involution. If g is odd then the fixed points of $\varphi_{\mathbb{C}}^2$ are fixed points of $\varphi_{\mathbb{C}}^{g+1}$ and, since $e(Q_1) = 2$ and $e(Q_2) = g + 1$, we get $\mu(\varphi_{\mathbb{C}}^{g+1}) = 2g + 6$. It leads to a contradiction and we conclude that g is even. Since the map $\pi'' : X \rightarrow \mathbb{P}_{\mathbb{R}}^1 \simeq X/\langle\varphi^{g+1}\rangle$ is without real ramification points, X is given by the real polynomial equation $y^2 = f(x)$, where f is a monic polynomial of degree $2g + 2$ and where f has no real roots. By [G-H, Prop. 6.3] we have $s = 1$. \square

Remark 4.5 In the situation of Theorem 4.4, we have $\mu(\varphi_{\mathbb{C}}^2) = 4$ and $|\varphi^2| = g + 1 = g + 2 - s$. It demonstrates that the inequality given in Theorem 2.9 is sharp.

4.3 Automorphisms of prime order

We bound above the order of an automorphism φ of a real curve when $|\varphi|$ is prime.

Theorem 4.6 *Let φ be an automorphism of X of prime order p such that $\pi : X \rightarrow X/\langle\varphi\rangle$ is without real ramification points. Then $p \leq g + 1$.*

Proof: Let g' denote the genus of $X' = X/\langle\varphi\rangle$. Since p is prime, any ramification point P of $\pi_{\mathbb{C}}$ is a totally ramified point i.e. $e_P = p$ and P is a fixed point of $\varphi_{\mathbb{C}}$. Consequently, π does not have any real branch point i.e. $w_{\mathbb{R}}(\pi) = 0$.

Assume $p \geq g + 2$. By Proposition 2.11 we have $\mu(\varphi_{\mathbb{C}}) \leq 2$.

If $\varphi_{\mathbb{C}}$ is fixed point free, it follows from (2) that $p = \frac{g-1}{g'-1}$ with $g' \geq 2$. Hence $p \leq g - 1$ which gives a contradiction.

If $\mu(\varphi_{\mathbb{C}}) = 2$ then we get $2g - 2 = p(2g' - 2) + 2(p - 1)$ by (2). Hence $p = \frac{g}{g'}$ and $g' \geq 1$. It gives a contradiction.

We have proved that $p \leq g + 1$. \square

4.4 The abelian case

We give now an upper bound of the order of an abelian group of automorphisms of a real curve.

Theorem 4.7 *Let G be an abelian group of automorphisms of a real curve X such that $\pi : X \rightarrow X' = X/G$ is without real ramification points.*

(i) *If $w(\pi_{\mathbb{C}}) = 0$ then $|G| \leq g - 1$.*

(ii) If $w(\pi_{\mathbb{C}}) > 0$ then $|G| \leq g + 3 + 2(g + 1 - s) \leq 3g + 3$.

Proof: Let g' denote the genus of X' . We denote by s' the number of connected components of $X'(\mathbb{R})$. If $w(\pi_{\mathbb{C}}) = 0$ the Riemann-Hurwitz relation (2) gives $|G| \leq g - 1$.

Assume $w_{\mathbb{C}}(\pi) = 0$ and $w_{\mathbb{R}}(\pi) > 0$. Since $\pi : X \rightarrow X' = X/G$ is without real ramification points, we have $\text{Int}(\pi) = 0$ and it follows that $s' \geq 2$, hence that $g' \geq 1$ by Harnack inequality. By Lemma 4.1, $w_{\mathbb{R}}(\pi) > 1$. Let Q_1, Q_2 be two distinct real branch points of π and let P_1 be a point in the fiber $\pi_{\mathbb{C}}^{-1}(Q_1)$. By [Pa, Lem. 1.1], $\text{Stab}(P_1)$ is cyclic. Let φ_1 be a generator of $\text{Stab}(P_1)$. We have $|\varphi_1| = e(Q_1)$. Since G is abelian, all the points in the fiber $\pi_{\mathbb{C}}^{-1}(Q_1)$ are fixed points of $\varphi_{1,\mathbb{C}}$. By (6) and since there are $\frac{|G|}{e(Q_1)}$ points in the fiber $\pi_{\mathbb{C}}^{-1}(Q_1)$, we obtain

$$\frac{|G|}{e(Q_1)} \leq 4 + 2\frac{g+1-s}{e(Q_1)-1} \leq 4 + 2(g+1-s). \quad (14)$$

Similarly, we have $\frac{|G|}{e(Q_2)} \leq 4 + 2(g+1-s)$. By (2) and according to the above remarks, we obtain $2g - 2 \geq (2g' - 2)|G| + \sum_{i=1}^2 (|G| - \frac{|G|}{e(Q_i)}) \geq 2|G| - 8 - 4(g+1-s)$. Hence $|G| \leq g + 3 + 2(g+1-s)$.

Assume $w_{\mathbb{C}}(\pi) > 0$. Let Q_1, \bar{Q}_1 be two conjugate branch points of $\pi_{\mathbb{C}}$. Arguing as in the previous case, we get

$$\frac{|G|}{e(Q_1)} = \frac{|G|}{e(\bar{Q}_1)} \leq 2 + (g+1-s). \quad (15)$$

If $g' \geq 1$ then, combining (2) with (15), we get $2g - 2 \geq (2g' - 2)|G| + 2(|G| - \frac{|G|}{e(Q_1)}) \geq 2|G| - 4 - 2(g+1-s)$ i.e. $|G| \leq g+1+(g+1-s)$. Let us assume that $g' = 0$. Since $s' = 1$ and $\pi : X \rightarrow X' = X/G$ is without real ramification points, we have $w_{\mathbb{R}}(\pi) = 0$. By (2), we see that $w_{\mathbb{C}}(\pi) \geq 2$ and let Q_2, \bar{Q}_2 be two conjugate branch points of $\pi_{\mathbb{C}}$ distinct from Q_1, \bar{Q}_1 . Clearly, we also have $\frac{|G|}{e(Q_2)} = \frac{|G|}{e(\bar{Q}_2)} \leq 2 + (g+1-s)$. The Riemann-Hurwitz relation (2) gives $2g - 2 \geq -2|G| + 2 \sum_{i=1}^2 (|G| - \frac{|G|}{e(Q_i)}) \geq 2|G| - 8 - 4(g+1-s)$ i.e. $|G| \leq g+3+2(g+1-s)$. \square

In case of an M-curve, the previous theorem reads:

Corollary 4.8 *Let G be an abelian group of automorphisms of an M-curve X such that $\pi : X \rightarrow X' = X/G$ is without real ramification points.*

(i) If $w(\pi_{\mathbb{C}}) = 0$ then $|G| \leq g - 1$.

(ii) If $w(\pi_{\mathbb{C}}) > 0$ then $|G| \leq g + 3$.

4.5 The hyperelliptic case

Before giving an upper bound on the order of the automorphisms group of a real hyperelliptic curve such that the hyperelliptic involution is without real fixed point, we prove a more general result.

Theorem 4.9 *Let X be a real curve such that $\pi : X \rightarrow X' = X/\text{Aut}(X)$ is without real ramification points. Assume there exists $\varphi \in Z(\text{Aut}(X))$, $\varphi \neq \text{Id}$, such that $\mu(\varphi_{\mathbb{C}}) > 0$, then $|\text{Aut}(X)| \leq 2g + 2 + 2\frac{g+1-s}{|\varphi|-1} \leq 2g + 2 + 2(g + 1 - s)$.*

Proof: Let g' denote the genus of X' . We denote by s' the number of connected components of $X'(\mathbb{R})$. Set $N = |\text{Aut}(X)|$.

Let P be a fixed point of $\varphi_{\mathbb{C}}$, then \bar{P} is also a fixed point of $\varphi_{\mathbb{C}}$.

Firstly, assume $Q = \pi_{\mathbb{C}}(P)$ is a real point. Hence $Q = \pi_{\mathbb{C}}(\bar{P})$. Since $\pi : X \rightarrow X' = X/\text{Aut}(X)$ is without real ramification points, we have $\text{Int}(\pi) = 0$ and it follows that $s' \geq 2$, hence that $g' \geq 1$ by Harnack inequality. By [Fa-K, Cor. p. 100], $\text{Stab}(P)$ is cyclic. Since the points in the fiber above Q have conjugate stabilizer subgroups and since $\varphi \in Z(\text{Aut}(X))$, the fiber $\pi_{\mathbb{C}}^{-1}(Q)$ is composed by $\frac{N}{e(Q)}$ points which are fixed points of $\varphi_{\mathbb{C}}$. By (6), we obtain $\frac{N}{e(Q)} \leq \mu(\varphi_{\mathbb{C}}) \leq 4 + 2\frac{g+1-s}{|\varphi|-1} \leq 4 + 2(g + 1 - s)$. By (2) and according to above remarks, we obtain $2g - 2 \geq (2g' - 2)N + (N - \frac{N}{e(Q)}) \geq N - \frac{N}{e(Q)}$. Hence $N \leq 2g - 2 + \frac{N}{e(Q)} \leq 2g + 2 + 2\frac{g+1-s}{|\varphi|-1} \leq 2g + 2 + 2(g + 1 - s)$.

Finally, assume $Q = \pi_{\mathbb{C}}(P)$ is a non-real point. Hence $\bar{Q} = \pi_{\mathbb{C}}(\bar{P})$ i.e. P and \bar{P} are contained in two conjugate fibers of $\pi_{\mathbb{C}}$. Arguing as in the previous case, we get $\frac{N}{e(Q)} = \frac{N}{e(\bar{Q})} \leq 2 + \frac{g+1-s}{|\varphi|-1}$. If $g' \geq 1$, it follows from (2) that $2g - 2 \geq (2g' - 2)N + 2(N - \frac{N}{e(Q)}) \geq 2N - 4 - 2\frac{g+1-s}{|\varphi|-1}$ i.e. $N \leq g + 1 + \frac{g+1-s}{|\varphi|-1}$. Let us assume that $g' = 0$. By (2), we see that $w_{\mathbb{C}}(\pi) \geq 2$ and let Q_1, \bar{Q}_1 be two conjugate branch points of $\pi_{\mathbb{C}}$ distinct from Q and \bar{Q} (the branch points are non-real since $g' = 0$). Clearly, $e(Q_1) = e(\bar{Q}_1) \geq 2$. The Riemann-Hurwitz relation (2) gives $2g - 2 \geq -2N + 2(N - \frac{N}{e(Q)}) + 2N(1 - \frac{1}{e(Q_1)}) \geq N - 4 - 2\frac{g+1-s}{|\varphi|-1}$ i.e. $N \leq 2g + 2 + 2\frac{g+1-s}{|\varphi|-1}$, which completes the proof. \square

Corollary 3.9 and the following proposition proves that the order of the automorphisms group of a real hyperelliptic curve cannot be larger than $4g + 4$.

Proposition 4.10 *Let X be an hyperelliptic curve such the hyperelliptic involution ι does not have any real fixed point. If $X' = X/\text{Aut}(X)$ is without real ramification points then either g is odd and $|\text{Aut}(X)| \leq 4g$, or $|\text{Aut}(X)| \leq 4g + 2$. If $X' = X/\text{Aut}(X)$ has at least one real ramification point then $|\text{Aut}(X)| \leq 4g + 4$.*

Proof: By [Fa-K, Cor. 3 p. 102], the hyperelliptic involution ι is in the center of $\text{Aut}(X)$. Let g' denote the genus of X' . In the case $\pi : X \rightarrow X' = X/\text{Aut}(X)$ is without real ramification points, the proof follows from Theorem 4.9 and from [Mo, Prop. 4.3].

Now we consider the case when $\pi : X \rightarrow X' = X/\text{Aut}(X)$ has real ramification points and ι does not have real fixed points. By Proposition 2.1 (iv), there exists two real branch points Q_1, Q_2 of π with real fibers and we have $e(Q_1) = e(Q_2) = 2$. Let P be a fixed point of ι . Since the points in the fiber above $Q = \pi_{\mathbb{C}}(P)$ have conjugate stabilizer subgroups and since $\iota \in Z(\text{Aut}(X))$, the fiber $\pi_{\mathbb{C}}^{-1}(Q)$ is composed by $\frac{N}{e(Q)}$ points which are non-real fixed points of ι . If $e(Q) = 2$ then $\frac{N}{2} \leq \mu(\iota) = 2g + 2$ and we get $N \leq 4g + 4$. So we assume $e(Q) \geq 4$ ($\text{Stab}(P)$ has even order since it contains ι). If $g' \geq 1$, from (2), we have $2g - 2 \geq (2g' - 2)N + N(1 - \frac{1}{e(Q)}) + N(1 - \frac{1}{e(Q_1)}) + N(1 - \frac{1}{e(Q_2)}) \geq \frac{3N}{4} + \frac{N}{2} + \frac{N}{2}$, hence

$N \leq \frac{8}{7}(g-1) \leq 2g-2$. Let us assume that $g' = 0$. By (2), we have $w(\pi_{\mathbb{C}}) > 3$. Let Q' be another branch point of $\pi_{\mathbb{C}}$ distinct from Q_1, Q_2 and Q . By (2) and since $\frac{N}{e(Q')} \leq \frac{N}{2}$, we have $2g-2 \geq (2g'-2)N + N(1 - \frac{1}{e(Q)}) + N(1 - \frac{1}{e(Q')}) + N(1 - \frac{1}{e(Q_1)}) + N(1 - \frac{1}{e(Q_2)}) \geq -2N + \frac{3N}{4} + \frac{N}{2} + \frac{N}{2} + \frac{N}{2}$, hence $N \leq 8g-8$. If $\pi_{\mathbb{C}}^{-1}(Q)$ is the unique fiber of $\pi_{\mathbb{C}}$ containing a fixed point of ι then $\frac{N}{e(Q)} = \mu(\iota) = 2g+2$, which contradicts the inequality $N \leq 8g-8$. Therefore, we may assume that $\pi_{\mathbb{C}}^{-1}(Q')$ is composed by $\frac{N}{e(Q')}$ points which are non-real fixed points of ι . Similarly, we assume $e(Q') \geq 4$. By (2), we get $2g-2 \geq (2g'-2)N + N(1 - \frac{1}{e(Q)}) + N(1 - \frac{1}{e(Q')}) + N(1 - \frac{1}{e(Q_1)}) + N(1 - \frac{1}{e(Q_2)}) \geq -2N + \frac{3N}{4} + \frac{3N}{4} + \frac{N}{2} + \frac{N}{2}$ i.e. $N \leq 4g-4$. \square

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